The holographic principle is a simple consequence of the divergence theorem

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Abstract

We define information as the ability to distinguish something from nothing, and show that the basic information unit corresponds to the surface area of a Planck sphere (a sphere with radius equal to the Planck length). Information in a given volume consequently may be encoded on the volume's bounding surface as a binary sequence or a Fourier series. Having established the holographic principle for information, we then use the divergence theorem to show the principle is general. The bit capacity of a gravitational field on the surface of a sphere of mass M is equal to the number of Planck masses contained in M. Hawking-Beckenstein black hole entropy and Beckenstein's generalized entropy bound for a matter system are re-expressed in the new information unit. Finally, the results are extended to define minimal time units ('something occurs') as the occasion of first passage of a stochastic proceess into or out of a Planck sphere. Time is shown to correspond to a probability distribution in space: at the Planck scale there is no 'spacetime', only space. Some speculation is offered for 4D space.

The holographic principle arose out of t'Hooft's [5] belief that in order to unify quantum mechanics with gravity, it is necessary to reduce 3D space to a 2D surface, much like a hologram which encodes a section of 3D space as a light interference pattern on a 2D surface. Susskind elaborates: "In a certain sense the world is two dimensional and not three dimensional as previously supposed" [11], while Ng and Van Dam sucinctly state: "In essence, the holographic principle says that although the world around us appears to have three spatial dimensions, its contents can actually be encoded on a two-dimensional surface" [10]. Bousso [3] in his survey relates the holographic principle to a 'covariant entropy bound', but notes the latter "may still prove incorrect or merely accidental".

The present article shows the holographic principle may be derived from simple information considerations, and the generality of this derivation is ensured by the divergence theorem of vector calculus.

1 Information: the ability to distinguish something from nothing

Surely one of the most elementary notions of information is the ability to distinguish something from nothing. If something can't be distinguished from nothing, then our system discriptions aren't going to be very reliable. An inability to distinguish something from nothing occurs near the Planck scale. Recall in this regard the Planck time:

$$t_P = \sqrt{\frac{\hbar G}{c^5}} \cong 5.39121 \times 10^{-44} s \tag{1}$$

where $\hbar$ is Planck's constant h divided by 2π, G is Newton's gravitational constant, and c is the speed of light. The Planck length is

$$l_P = ct_P = \sqrt{\frac{\hbar G}{c^3}} \cong 1.61624 \times 10^{-33} cm, \tag{2}$$

and the Planck energy is

$$E_P = \frac{\hbar}{t_P} = \sqrt{\frac{\hbar c^5}{G}} \cong 1.9561 \times 10^9 joules \cong 1.22090 \times 10^{28} eV. \tag{3}$$

It is important to realize that at the Planck scale, not only does 'spacetime' break down, in some sense, but that matters are much 'worse' than that. At that level, matter and vacuum can't be distinguished.

To see this, suppose we have a tiny sphere of radius L which we choose to fill with the maximum mass possible, or with the minimum mass possible (a vacuum). The maximum mass possible is given by a black hole with the same radius as the sphere. The minimum mass possible is, due to Heisenberg's indeterminacy relation, given by the mass whose Compton wavelength fits on the surface of the sphere. Now, the Schwarzchild radius for a black hole for an object of mass M is $r_S = \frac{2MG}{c^2}$. If the sphere has uncertainty in radius $\frac{\hbar}{Mc}$, then the uncertainty in the length of a wave sitting on the surface of the sphere is $\lambda_C = \frac{2\pi\hbar}{Mc}$. If we set $r_S = \lambda_C = L$, then inside the sphere we have a mass M with the constraints

$$\frac{c^2 L}{2G} \geq M \geq \frac{2\pi\hbar}{cL}. \tag{4}$$

If we substitute the Planck length l_P for L in inequality Eq.(4), we discover that the lower bound exceeds the upper bound. So we will have lost our ability to distinguish something from nothing before L reaches the Planck length l_P. We would not be able to tell whether the sphere contained a black hole or vacuum. Matter and vacuum would appear to be the same stuff. Information would not exist in this environment.

Let's solve for the value of L that allows for information. Eq.(4) implies

$$L^2 \geq 4\pi \frac{\hbar G}{c^3} = 4\pi l_P^2. \tag{5}$$

Now for a Planck sphere of radius $r = l_P$, the surface area $A_P = 4\pi l_p^2$. Hence we have

$$L^2 \geq A_P. \tag{6}$$

We will take the minimal value of L^2 as our *elementary pixel of information.* This pixel of information $L_m^2 = A_P$ has area equal to the surface area of a Planck sphere. At this level and above we can distinguish something from nothing. Thus, for example, we can encode a binary 0 or a 1 in an information pixel. The value is 1 if it contains 'something' and 0 if it contains 'nothing'. Below the surface area of a Planck sphere we can no longer do even binary computation, because we cannot distinguish a 1 from a 0. Thus *area* here corresponds directly to *information.* Thus we define our elementary unit of information I_0 as

$$I_0 = L_m^2 = A_P = \text{surface area of a Planck sphere.} \tag{7}$$

Since any volume can be measured in terms of the volume of a Planck sphere ($\frac{4}{3}\pi l_P^3$), and the boundary surface in terms of the surface area of a

Planck sphere ($4\pi l_P^2$), we have essentially established the holographic principle simply from the non-abitrary way we have defined information. The total information I is proportional to the volume V raised to the $\frac{2}{3}$ power:

$$I \propto V^{\frac{2}{3}}. \tag{8}$$

Note that Susskind's [11] number of states $N(V) - 2^n$ in a volume V should be corrected to read

$$\ln N(V) = n \ln 2 = \frac{3V}{4\pi} \frac{\ln 2}{l_P^3}, \tag{9}$$

because Susskind considers a volume too small for 0s and 1s to be distinguished: l_P^3 instead of $\frac{4}{3}\pi l_P^3$.

If we consider a volume V defined by bounding surface area divided into information pixels of size A_P, we can then define the information in the volume by a binary sequence. For example, consider a 3×3 square area, where, reading from upper left to lower right, the value of the pixels is

$$100110101. \tag{10}$$

This encodes the information contained in the volume. In addition, we can consider a sequence like Eq.(10) as a 'time series' or data sequence: $x_0 = 1$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, etc. In general, the time series or data sequence x_t: $x_0, x_1, x_2, \cdots, x_{n-1}$, can be represented as [2]

$$x_t - A_0 + \sum_{0<j<n/2} (A_j \cos\, \omega_j t + B_j \sin\, \omega_j t) + (-1)^t A_{\frac{n}{2}}. \tag{11}$$

Here $\omega_j = 2\pi j/n$ is the j-th Fourier frequency. The other paramters are calculated as

$$A_0 = \frac{1}{n} \sum_{t=0}^{n-1} x_t \tag{12}$$

$$A_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t \cos\, \omega_j t \tag{13}$$

$$B_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t \sin\, \omega_j t \tag{14}$$

for $0 < j < n/2$. If n is even, then

$$A_{\frac{n}{2}} = \frac{1}{n} \sum_{t=0}^{n-1} (-1)^t x_t, \tag{15}$$

else $A_{\frac{n}{2}} = 0$. Eq.(11) defines a 'hologram', a superposition of waves, on which the information in the volume V is encoded.

2 The divergence theorem and the information capacity of a gravitational field on a sphere

We have established the holographic principle for information I, but we might ask the question: Is this an artifact of the definition of information, no matter how compelling that definition is? The divergence theorem indicates not; rather, the principle is general.

Consider a vector field F defined over a volume $V \subset R^3$ with infinitesimal volume element dV. Let the volume be enclosed by surface S, with infinitesimal element of area dS. Let $\mathbf{n}$ be a unit normal to the surface S. The *divergence theorem* says

$$\int\int\int \nabla \cdot F \; dV = \int\int F \cdot \mathbf{n} \; dS. \tag{16}$$

The divergence theorem relates a volume integral to the integral over the area of the enclosing surface. (The multiple integrals are usually simplified to a single integral sign.)

An illustration of the divergence theorem that is perhaps most familiar is Gauss's Law, which says that if there is a charge Q in a volume, and D is the electric displacement vector on the enclosing surface S of the volume, then

$$Q = \oint \mathbf{D} \cdot \mathbf{n} \; dS. \tag{17}$$

But the application of the divergence theorem is quite general, and the same mathematics applies to electric and magnetic fields, or to the flow out of a volume of fluid, heat, gas, or a stream of particles from a radioactive source. What happens within an enclosed volume is reflected on the enclosing surface, and this is the holographic principle in a nutshell. Thus it is no surprise that our basic information pixel $A_P = 4\pi l_P^2$ turned out to be the surface area of a Planck sphere.

Consider a mass M distributed inside a sphere. Then, regardless of the distribution, the absolute value of the surface integral is equal to $4\pi GM$. We can convert this result to proper units of area (meters^2) by multiplying by $\frac{t_P}{c}$, and then obtain the number of basic information units by dividing by I_0:

$$\text{information bits in gravity field of sphere of mass} M = I_{GM} = \frac{4\pi MG}{I_0}\frac{t_P}{c} \quad (18)$$

$$= \frac{MG}{l_P^2}\frac{t_P}{c}. \quad (19)$$

If we now substitute $G = \frac{l_P^3}{m_P t_P^2}$, where m_P is the Planck mass ($m_P \cong 2.17645 \times 10^{-8}$ kg), we obtain the number of bits as

$$\text{information bits in gravity field of sphere of mass} M = \frac{M}{m_P}. \quad (20)$$

A gravity field on a sphere of mass M contains information bits equal to the number of Planck masses m_P contained in M. If we consider the sphere as a quantum computer, then it has a memory capacity of $\frac{M}{m_P}$ bits. Each Planckian mass yields a bit, a 0 or a 1, in the binary representation of the gravitational field resulting from the mass in the sphere/computer, hence converting 3D to 2D. The divergence theorem thus gives us the holographic principle.

3 Hawking-Beckenstein black hole entropy and the Beckenstein generalized entropy bound

We can compare these results with the Hawking-Beckenstein result for the entropy S_B of a black hole. According to them (see references in [7]), the entropy of a non-rotating, uncharged black hole with horizon area A_{hor} is

$$S_B = \frac{k_b \ln 2\ c^3}{8\pi G\hbar} A_{hor} \quad (21)$$

where k_b is Boltzman's constant. This can be rewritten

$$S_B = \frac{k_b \ln 2}{2(4\pi l_P^2)} A_{hor} = \frac{1}{2} k_b \ln 2\ \frac{A_{hor}}{I_0}. \quad (22)$$

Now $\frac{A_{hor}}{I_0}$ is simply the horizon surface area measured in elementary information units, or pixels, I_0. So $\frac{A_{hor}}{I_0}$ is the number of bits of information. We thus have the usual result of quantum entropy being the number of bits multiplied by ln 2, but with the additional factor of $\frac{1}{2}k_b$.

The generalized Beckenstein entropy bound for a matter system is

$$S_{\text{matter}} \leq \frac{2\pi k_b E R}{\hbar c}, \tag{23}$$

where R is the radius of the smallest sphere that fits around the matter system, and E is the mass-energy. This may be rewritten as

$$S_{\text{matter}} \leq \frac{4\pi k_b M c^2 R}{2m_P^2 G} = 4\pi k_b (\frac{R}{r_{_{PB}}})(\frac{M}{m_P}), \tag{24}$$

where $r_{_{PB}} = \frac{2m_P G}{c^2}$ is the Schwarzchild radius of a Planckian black hole. Thus Beckenstein's generalized matter entropy is less than or equal to the number of bits of information $(\frac{M}{m_P})$ multiplied by the radius of the enclosing sphere measured in Planckian black hole radii $(\frac{R}{r_{_{PB}}})$ multipled by $4\pi k_b$.

Furthermore, we can measure Beckenstein's generalized entropy bound in terms of the entropy of a black hole. We obtain

$$\frac{S_{\text{matter}}}{S_B} \leq \frac{8\pi}{\ln 2} \frac{R}{r_{_{PB}}} \frac{\text{number of bits in matter system}}{\text{number of bits in the black hole}}. \tag{25}$$

Thus Beckenstein's generalized entropy, measured in terms of black hole entropy, is proportion to the radius R of the enclosing sphere measured in Planckian black hole radii $r_{_{PB}}$, multiplied by the number of bits in the matter-system relative to the number of bits in the black hole.

4 Time units: something occurs

Consider now minimal time units. These are not the Planck time t_P. Events taking place in the interior of a Planck sphere are not observable. They become observable only when they appear on or outside the surface of the Planck sphere. This leads us to consider exit distributions. In this regard, imagine a stochastic process taking place within the Planck sphere. In particular, a symmetric stable process $[X(t), t \geq 0]$. Since there is no 'time' inside the Planck sphere, 't' should be considered a purely formal parameter that indexes the process. The process has stationary, independent increments with transition density in R^N

$$s(t,x) = (2\pi)^{-N} \int e^{i(x\cdot\epsilon)} e^{-t|\epsilon|^\alpha} d\epsilon. \tag{26}$$

with characteristic exponent α. Here x and ϵ are points in R^N, and $X(0) = x$ with probability one. For $\alpha = 2$, this is the familiar Wiener-Lévy process, which is integrable, nowhere differentiable, and has continuous sample paths. For $\alpha < 2$, the sample paths are generally more cohesive than in a Wiener-Lévy process, but are however punctuated by discontinuous jumps [9]. The discontinuous jumps means the stable process can exit the sphere to, with positive probability, any point in space outside the sphere. It does not have to reside on the boundary of the sphere in the process.

Let $l_P - 1$. That is, we measure everything in units equal to the Planck length, and hence the Planck sphere becomes the unit sphere. Let T be the occasion of first passage out of the Planck sphere, which also can be considered as the flipping of a 0 to a 1. Let T^* be the occasion of first passage into the Planck sphere from a point x outside the sphere, which also can be considered as the flipping of a 1 to a 0:

$$T = \inf \{t|X(t) > 1\} \tag{27}$$
$$T^* = \inf \{t|X(t) < 1\}. \tag{28}$$

This sets the 'clock rate' of a 0 changing to 1 as the expected value of T, $< T >$, while the rate of a 1 changing to 0 is the expected value of T^*, $< T^* >$. Note that this process of 0s flipping to 1s, or 1s to 0s, along with the possible asymmetry of flip rates, is reminiscent of the collapse to 0s or 1s in the vortice solutions to the imaginary part of Schrödinger's wave equation [4]. If the rate of 1s flipping to 0s is slower, the number of 1s must increase relative to 0s until the numbers of flips in either direction are equalized. That means matter must grow in the universe until flip rates are equalized. The growth of the universe comes to a halt, however, once sufficient 1s are in existence.

These minimal time units are related to a probability distribution in space. Define $f(x,y)$ as

$$f(x,y) = \pi^{-(\frac{N}{2}+1)} \Gamma(\frac{N}{2}) \sin \frac{\pi\alpha}{2} |1 - |x|^2|^{\alpha/2} |1 - |y|^2|^{-\alpha/2} |x-y|. \tag{29}$$

Then, at $t = T$, the probability density of the symmetric stable process $X(t)$ beginning at x inside the Planck (unit) sphere being found at y outside the sphere is, for $0 < \alpha < 2$, [1]

$$f(x, y)dy. \tag{30}$$

Similarly, at $t = T^*$, the probability density of the symmetric stable process $X(t)$ beginning at x outside the Planck (unit) sphere being found at y inside the sphere is, for $\alpha < N$ or for $N = \alpha = 1$,

$$f(x, y)dy. \tag{31}$$

This defines the minimal time unit, $< T >$ or $< T^* >$, as a probability distribution in space. Notice some implications of this result.

Consider a simplistic and purely heuristic picture of a 'quantum foam' which consists of a Planck sphere containing a symmetric stable process floating in a 'vacuum'. To observation, there is only vacuum. Then, in an observed region of space, 'something occurs'. A 0 flips to a 1. Then according to Eq.(29), we can say that with higher probability this was due to exit of the process from a Planck sphere which is nearby. But there is always some probability that the Planck sphere is distant. In other words, local effects are not necessarily due to local causes. This would seem to require a revision to 'quantum causal histories/networks', such as examined in [8].

In addition, since 'time' as defined by the clock rate here simply corresponds to a probability distribution of matter in space, time is in no sense a separate dimension. There is no spacetime, but only space with its distribution of matter. Thus the Minkowski metric breaks down at the Planck scale.

The minimal time units, $< T >$ and $< T^* >$, define cut-off frequencies, $\frac{1}{<T>}$ and $\frac{1}{<T^*>}$, which are considerably smaller than the cut-off frequency, $\frac{1}{t_P}$, implied by Planck time.

5 Possible implications for 3D space

Susskind's statement that "in a certain sense the world is two dimensional and not three dimensional as previously supposed" suggests another possibility. Rather than considering ourselves flatlanders, or 2D shadows on Plato's cave, it is equally plausible that we live in 3D as commonly supposed, but that this 3D space is the enclosing surface of a 4D space (*not* spacetime). If so, then the divergence theorem indicates that we can gain a 'black and white' version of what is happening in 4D space. Events in 3D space may in some cases have their origin in 4D, outside our Universe. To determine this, we would need to look for 3D phenomena that are not fully explainable in 3D–including possibly such things as the origin of the Big Bang, quantum uncertainty, or even the human mind.

The 0 and 1 nature of our basic information pixel can be considered alternatively as 'black' or 'white', which as 't Hooft [5] notes represents the loss of color and hence information in a hologram. The flux out of the interior of 4D space must impinge on the 3D surface, but not in its original technicolor. However, events in 4D space should be detectable in some way. Perhaps more importantly, observed 3D events cannot be adequately interpreted without reference to the underlying 4D space that is not directly observed.

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